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Subject-Coincident Coordinate Systems and Sustained Motions

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Vestibular research on human perception of self-motion and orientation generally uses the head-based coordinate system standardized by Hixson, Niven, and Correia (1966) for specifying accelerations of the subject. This paper expands the head-based system to include velocities, thereby incorporating both the visual and vestibular systems, and formally defines the resulting concept of a *subject-coincident coordinate system*. By capturing the organism's vantage point during self-motion, subject-coincident systems give a natural framework for studying the relationship between stimulus, physiology, and perception; however, the essential approach differs from that familiar in traditional physics, so the necessary equations of motion are developed here. In addition, these equations are used to investigate the set of *sustained motions*, those motions that can be sustained over a period of time. These motions can cause disorientation and misperception of motion because of saturation or adaptation of the human sensory receptors. The results on sustained motions are summarized in a complete categorization of the set of sustained motions.

1. INTRODUCTION

A physicist rarely asks a moving particle, "How do you feel during this motion?" We do not expect the physicist to ask such a question, as long as the physics explains the overall motion of the particle. On the other hand, researchers of self-motion perception are quite interested in how a subject feels and how the sensory systems are stimulated during a given motion of the subject. While standard equations from physics and engineering are useful for describing and analyzing a subject's overall motion, a description and analysis of the motion from the *subject's* point of view requires equations and mathematical structures custom-made for research in neuroscience.

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Neurophysiology often deals with a stimulus to the organism, the stimulus' effect on the sensory receptors, and the ensuing effects within the central nervous system. The stimulus in an experiment is usually simple—pressure on the skin, an auditory tone, a flash of light, linear acceleration in a straight line—allowing the scientist to focus more attention on the succeeding neural events. However, the stimulus of motion is, by nature, multidimensional and complex. As we study natural movement of humans and other organisms, complexity creeps up on us from all directions, in the three dimensions of velocity, the three dimensions of acceleration, and especially in the interrelationships between the various components of motion. A thorough understanding of the stimulus, however complicated, is necessary in our exploration of the sensing, perceiving organism.

Studies of self-motion perception are numerous, as can be seen in reviews by Guedry (1974, 1992) and Dichgans and Brandt (1978). NASA and the Armed Forces are particularly interested in understanding self-motion perception because of the unusual conditions encountered during air- and spaceflight. Many aircraft accidents occur each year due to pilot disorientation. One example of a disorienting and dangerous situation is the graveyard spiral, in which the airplane travels in a coordinated turn (meaning that the pilot feels no forces to the left or right) while descending. If the pilot interprets the sensory cues as signaling descent with no turn, he or she may pull back on the stick to stop the descent. However, this will pull the airplane into a tighter spiral and a faster descent. Disorientation and misperception of motion occur not only in the graveyard spiral, but also in the graveyard spin, the leans, and many other situations in flight (see, for example, Peters, 1969).

Misperception of motion is obviously dangerous in an airplane, but it also has consequences in other forms of modern transportation such as cars, boats, buses, bicycles, trains, etc., and even when walking, standing, or sitting. Our sense of balance is something we often take for granted, but many persons with vestibular disorders deal with a constant struggle to determine who is moving: "Am I starting to fall forward, or is my friend starting to lean toward me?" A better understanding of human sensory systems and the central nervous system's "production" of perception would be helpful to clinical medicine.

Physiologically, the vestibular system of the inner ear is known to play a major role in self-motion perception. With the three approximately orthogonal semicircular canals on each side of the head, all directions of angular acceleration in three dimensions are detected, and with the pair of otolith organs on each side, all directions of linear acceleration are detected. The visual system also plays a major role in motion perception, with its ability to detect all directions of velocity, both angular and linear. Other sensory systems, such as the somatosensory and auditory systems, also contribute.

Toward an understanding of the physiological and psychological processes involved, research has analyzed perception of self-motion during many different kinds of motion. A simple example is on-axis rotation with a vertical axis. In particular, studies have shown that during sustained constant rotation on a rotating chair in the dark, a subject will feel stationary despite the continuing rotation. The sensation of stationarity may begin anywhere from a few seconds to a few minutes after the chair has reached constant velocity, depending on the startup acceleration; see Guedry (1974) for a review of the literature. Various studies of this nature have been done, each examining perception during a particular kind of motion.

The stimuli in such experiments have often been simple, but research is heading in the direction of more and more complicated motions (see, for example, Guedry *et al.*, 1992). For this reason, a solid grasp of the basic description of motion is necessary, especially as applies to perception. Hixson *et al.* (1966) recognized the necessity of a standard by which to describe the motion of a subject, and they provide a useful standard for specifying the linear and angular accelerations in a coordinate system oriented with the head [also described in Guedry (1974)]. This system has since been used by many researchers; however, Hixson *et al.* (1966) concentrate on the vestibular system, and therefore discuss accelerations, but not velocities. Naturally, velocity is equally important in the description of human movement when vision is available, so this paper includes both velocities and accelerations in a description of human movement.

We present the notion of a *subject-coincident coordinate system* in which velocities and accelerations, as well as orientation relative to the gravitational field, can be measured; these subject-coincident systems incorporate the fact that the sensory receptors of a subject move with the subject. While certain coordinate systems familiar in traditional physics and engineering have selected features in common with subject-coincident systems, we define subject-coincident systems explicitly to embody those properties relevant to the study of human self-motion perception, and develop the equations relating different components of motion as perceived by the moving subject.

With the foundation of subject-coincident coordinate systems, we then demonstrate a complete categorization of *sustained motions*. These are motions that can be sustained over time, the exact definition being given in Section 3. A constant-velocity version of the pilots' graveyard spiral is a sustained motion, as are rotation in a rotating chair, flying straight at 800 km/hr in a passenger jet, and standing stationary on the ground. One reason to study this class of motions is that they can cause the vestibular system to saturate or adapt in the sense that the peripheral receptor response levels off, not necessarily at the resting level, leading to misperceptions and disorientation. In fact, many sustained motions are perceptually indistinguishable. For example, a passenger in a jet during calm flight feels just as though the jet is not moving; even if the sound of the engines is used as a clue, the passenger could probably say at most, "I am moving," but not discern the speed of flight. While this example is quite simple, there are many complicated motions involving rotation and acceleration (such as the graveyard spiral) that are perceptually indistinguishable from other complicated, and even simple, motions. The development in the present paper of subject-coincident coordinate systems and the categorization of sustained motions form the foundation for further research on spatial disorientation; for example, a full classification of perceptually indistinguishable sustained motions is given in Holly and McCollum (1995), building upon the theory developed in the present paper.

2. SUBJECT-COINCIDENT COORDINATE SYSTEMS

Following the standard set by Hixson *et al.* (1966) (see also Guedry, 1974), self-motion is specified according to a coordinate system oriented with the head, as shown in Fig. 1. Forward and backward linear accelerations are along the x axis with forward being in the positive x direction, leftward and rightward linear accelerations are along the y axis with leftward being positive, while upward and downward are along the z axis with upward being positive. The convention of having unit vectors **i**, **j**, and **k** in the x, y, and z directions, respectively, is followed. Similarly, angular accelerations are specified in the xyz coordinate system, using the right-hand rule. In particular, rightward roll is given by a vector in the positive x direction, forward pitch is given by a vector in the positive y direction, and leftward yaw is given by a vector in the positive z direction. Even though these coordinates do not correspond exactly with the axes of the vestibular end organs, they serve

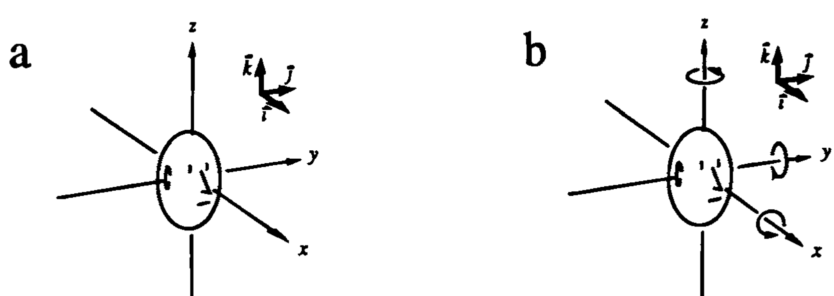

Fig. 1. Standard coordinates for specification of head motion. (a) Coordinates for linear motion, and standard unit vectors. (b) Coordinates for angular motion, and standard unit vectors.

as a means to discuss and relate various motions from the perspective of the subject.

Hixson *et al.* (1966) speak only of accelerations, but we include velocities (and thereby, the visual system, which can detect velocity relative to the environment) in our discussion. Linear and angular velocities of the subject are specified with the x, y, and z directions being the same as for accelerations (See Fig. 1).

This system specifies motion from the vantage point of the subject, so now it is possible to ask questions such as, "During a coordinated turn of an airplane, what is the motion description from the vantage point of the pilot?" The converse question is asked by the subject, "When I sense a forward linear velocity combined with z-directed linear acceleration and angular velocity about an off-z-axis, what is the airplane doing?" The answer to these questions and the manner in which humans handle questions such as the latter have a direct bearing not only on a pilot's ability to avoid disorientation, but also on every person's ability to balance and navigate during everyday movement.

In tackling the problem of the relationship between the head-based description of movement and an earth-based description of movement, we encounter the fact that acceleration is not the time derivative of velocity in the head system, and upon closer inspection, we see that the coordinate system is in some sense both moving and stationary. The coordinate system is moving with the head; however, linear velocity of the head is not necessarily zero with respect to the coordinate axes that are "fixed" in the head. More accurately, the coordinate axes are *coincident* with the head. In order to specify velocity or acceleration at a given point in time, the coordinate axes must stay fixed in position relative to the earth so that head movement can be measured within them. Of course, the coordinate axes can be fixed in position only momentarily since their position needs to coincide with that of the head at all times.[2]

Coordinate systems with similarities to that described above are encountered in traditional studies of rigid-body motion. In the standard physics literature (e.g., Goldstein, 1980), calculation of angular velocity of a spinning top using axes oriented with the top, for example, leads to the same value as would be measured using a top analog of the head coordinate system above. However, this calculation of angular velocity proceeds by way of angular momentum, and angular momentum is peripheral to the present needs; vision detects velocity, not momentum. In addition, we need a system that measures not only angular velocity, but also linear velocity in this unusual way.

[2] Another way to describe this situation is that motion is measured over time by using a continuum of distinct sets of earth-fixed coordinate axes coincident with the head.

Perhaps the coordinate systems most similar to ours are found in aeronautical engineering (e.g., Britting, 1971; Etkin, 1959), where linear velocity of an aircraft is sometimes described in terms of a nonzero vector specified in a coordinate system oriented with the aircraft.

The coordinate systems used in the present work capture the subject's vantage point during movement; for purposes of self-motion perception research, we define a *subject-coincident coordinate system* to be the natural generalization of the head coordinate system described above.

Definition 1. A *subject-coincident coordinate system* is given by a set of coordinate axes fixed in position relative to the object in question (such as the head), and for which movement of the object is specified in the following way: At each point in time, aspects of motion such as the linear velocity, linear acceleration, angular velocity, and angular acceleration of the object are given by their measurements in the (traditional) coordinate system whose axes are positioned with the coordinate axes of the object, but that are fixed in space relative to a predetermined "reference object" (such as the earth).

To investigate a subject's vantage point during self-motion and how it relates to an outside observer's viewpoint, equations relating these subject-coincident systems and traditional (e.g., earth-fixed) coordinate systems are useful. For completeness, and because our interests lie in components of motion such as velocity and acceleration rather than the commonly discussed aspects such as momentum and force, we derive from first principles the equations relevant to our purposes. In deriving the equations, we set the following conventions:

The object of interest is taken to be a subject's head, and "head-coincident" is often written in place of "subject-coincident," although the equations hold for any given object of interest, not just a head. The reference object is taken to be the earth. Note that small forces due to rotation of the earth are usually considered too subtle to be a factor in human sensory reception; however, this assumption is not strictly necessary here, and all equations and results in this paper hold even when the reference object has a noninertial reference frame. The vectors for velocities and accelerations of the head in the head-coincident coordinate system are given in units of measure and by notation as follows:

- Linear velocity in m/sec (meters/second) is denoted by ${}^{h}\mathbf{v}$
- Linear acceleration in m/sec^2 is denoted by ${}^{h}\mathbf{a}$
- Angular velocity in rad/sec (radians/second) is denoted by ${}^{h}\boldsymbol{\omega}$
- Angular acceleration in rad/sec^2 is denoted by ${}^{h}\boldsymbol{\alpha}$

Since each of these vectors has a value at each point in time, they are considered functions of time t and written as ${}^{h}\mathbf{v}(t)$, ${}^{h}\mathbf{a}(t)$, ${}^{h}\boldsymbol{\omega}(t)$, and ${}^{h}\boldsymbol{\alpha}(t)$, respectively. Those same vectors described in a predetermined coordinate system fixed relative to the earth, called the *earth-fixed coordinate system* here, are written as ${}^{E}\mathbf{v}(t)$, ${}^{E}\mathbf{a}(t)$, ${}^{E}\boldsymbol{\omega}(t)$, and ${}^{E}\boldsymbol{\alpha}(t)$, respectively. An uppercase superscript such as ^{E} is meant to indicate that the coordinate system is of the traditional type, while a lowercase superscript such as ^{h} is meant to signify a subject-coincident coordinate system. Vectors in each system are represented as column vectors for our calculations.

In keeping with the particular problem at hand, we assume the three axes of the head-coincident coordinate system are orthogonal, and that the three axes of the earth-fixed coordinate system are orthogonal.

A single linear transformation will perform the job of taking ${}^{h}\mathbf{v}$ to ${}^{E}\mathbf{v}$, ${}^{h}\mathbf{a}$ to ${}^{E}\mathbf{a}$, ${}^{h}\boldsymbol{\omega}$ to ${}^{E}\boldsymbol{\omega}$, and ${}^{h}\boldsymbol{\alpha}$ to ${}^{E}\boldsymbol{\alpha}$. This linear transformation is a rotation describing the orientation of the head-coincident coordinate axes relative to the earth-fixed coordinate axes. While three-dimensional rotations are sometimes described in terms of two or three successive rotations, the present three-dimensional rotation is most simply described as a single rotation (possible by Euler's theorem). For example, if the two coordinate systems start in alignment, and then the head rotates for 1 sec with constant angular velocity ${}^{E}\boldsymbol{\omega}$ (choosing earth-fixed coordinates because they give a fixed reference frame within which to specify a constant angular velocity), we say that the new *angular position* of the head axes is (1 sec)(${}^{E}\boldsymbol{\omega}$ rad/sec) = ${}^{E}\boldsymbol{\omega}$ rad.

In general, the angular position of the head described by rotation for t seconds at angular velocity ${}^{E}\boldsymbol{\omega}$ is ${}^{E}\boldsymbol{\omega}t$ rad. The symbol used here for angular position of the head with respect to the earth is $\boldsymbol{\xi}$, given in radians (rad).

Straightforward (but nontrivial) vector calculations reveal that the conversion from head-coincident coordinates to earth-fixed coordinates when the head axes are at nonzero angular position

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix}$$

relative to the earth-fixed axes is given by the linear transformation described by the matrix

$$R(\boldsymbol{\xi}) = \frac{1}{|\boldsymbol{\xi}|^2} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \tag{1}$$

where

$$
\begin{aligned}
s_{11} &= (|\boldsymbol{\xi}|^2 - \xi_x^2)\cos|\boldsymbol{\xi}| + \xi_x^2 \\
s_{12} &= -\xi_x\xi_y\cos|\boldsymbol{\xi}| - \xi_z|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_x\xi_y \\
s_{13} &= -\xi_x\xi_z\cos|\boldsymbol{\xi}| + \xi_y|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_x\xi_z \\
s_{21} &= -\xi_x\xi_y\cos|\boldsymbol{\xi}| + \xi_z|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_x\xi_y \\
s_{22} &= (|\boldsymbol{\xi}|^2 - \xi_y^2)\cos|\boldsymbol{\xi}| + \xi_y^2 \\
s_{23} &= -\xi_y\xi_z\cos|\boldsymbol{\xi}| - \xi_x|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_y\xi_z \\
s_{31} &= -\xi_x\xi_z\cos|\boldsymbol{\xi}| - \xi_y|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_x\xi_z \\
s_{32} &= -\xi_y\xi_z\cos|\boldsymbol{\xi}| + \xi_x|\boldsymbol{\xi}|\sin|\boldsymbol{\xi}| + \xi_y\xi_z \\
s_{33} &= (|\boldsymbol{\xi}|^2 - \xi_z^2)\cos|\boldsymbol{\xi}| + \xi_z^2
\end{aligned} \tag{2}
$$

When $\boldsymbol{\xi} = \mathbf{0}$, $R(\boldsymbol{\xi})$ is the identity matrix.

Since angular position $\boldsymbol{\xi}$ can change as a function of time, we write $\boldsymbol{\xi}(t)$ to denote the dependence on time t, and the matrix R becomes $R(\boldsymbol{\xi}(t))$.

The value of $\boldsymbol{\xi}(t)$ as t changes is not necessarily provided explicitly in a given real-life problem, so it may be computed from angular velocity by

$$\boldsymbol{\xi}(t) = \int_0^t {}^{\mathrm{E}}\boldsymbol{\omega}(\tau)\,d\tau + \boldsymbol{\xi}(0) \tag{3}$$

for all t whenever there exists $\mathbf{c} \neq \mathbf{0}$ with $\boldsymbol{\xi}(0) \parallel \mathbf{c}$ and ${}^{\mathrm{E}}\boldsymbol{\omega}(\tau) \parallel \mathbf{c}$ for all $\tau \in (0, t)$, i.e., whenever rotation occurs about a fixed axis. Similarly, ${}^{\mathrm{E}}\boldsymbol{\omega}$ may be computed from ${}^{\mathrm{E}}\boldsymbol{\alpha}$ by

$${}^{\mathrm{E}}\boldsymbol{\omega}(t) = \int_0^t {}^{\mathrm{E}}\boldsymbol{\alpha}(\tau)\,d\tau + {}^{\mathrm{E}}\boldsymbol{\omega}(0) \tag{4}$$

for all t whenever there exists $\mathbf{c} \neq \mathbf{0}$ with ${}^{\mathrm{E}}\boldsymbol{\omega}(0) \parallel \mathbf{c}$ and ${}^{\mathrm{E}}\boldsymbol{\alpha}(\tau) \parallel \mathbf{c}$ for all $\tau \in (0, t)$.

In more generality, if rotation occurs about a succession of axes, $R(\boldsymbol{\xi}(t))$ may be computed by a product depending on the angular velocity. For example, if there exist $\mathbf{c}_1$, $\mathbf{c}_2 \neq \mathbf{0}$ such that $\boldsymbol{\xi}(0) \parallel \mathbf{c}_1$ and ${}^{\mathrm{E}}\boldsymbol{\omega}(\tau) \parallel \mathbf{c}_2$ for all $\tau \in (0, t)$, then $R(\boldsymbol{\xi}(t)) = R(\int_0^t {}^{\mathrm{E}}\boldsymbol{\omega}(\tau)\,d\tau)R(\boldsymbol{\xi}(0))$. This equation can be extended to cover any finite sequence of rotations, but such an extension is not necessary for the investigation in this paper.

The most general case of rotation about a continuously varying axis can be analyzed using quaternions, but it suffices for present purposes to consider fixed axes of rotation. Therefore, we take advantage of the relatively simple equations (3) and (4) and reserve discussion of quaternions for future work.

Equations (5)–(13) below hold for all motions, including those with continuously varying axes of rotation. The vector ${}^{E}\boldsymbol{\omega}$ may be computed from ${}^{h}\boldsymbol{\omega}$,

$$ {}^{E}\boldsymbol{\omega}(t) = R(\xi(t))\ {}^{h}\boldsymbol{\omega}(t) \tag{5} $$

for all t. If angular acceleration is not known in earth-fixed coordinates, it can be calculated from ${}^{h}\boldsymbol{\alpha}$,

$$ {}^{E}\boldsymbol{\alpha}(t) = R(\xi(t))\ {}^{h}\boldsymbol{\alpha}(t) \tag{6} $$

for all t. Notice that equations (3) and (5) are coupled, as are (3), (4), and (6). Fortunately, our applications avoid the difficulties that this coupling might create.

Linear velocity in earth-fixed coordinates may be computed from ${}^{h}\mathbf{v}$,

$$ {}^{E}\mathbf{v}(t) = R(\xi(t))\ {}^{h}\mathbf{v}(t) \tag{7} $$

for all t, or from ${}^{E}\mathbf{a}$,

$$ {}^{E}\mathbf{v}(t) = \int_0^t {}^{E}\mathbf{a}(\tau)\, d\tau + {}^{E}\mathbf{v}(0) \tag{8} $$

for all t. If linear acceleration is not known in earth-fixed coordinates, it can be calculated from ${}^{h}\mathbf{a}$,

$$ {}^{E}\mathbf{a}(t) = R(\xi(t))\ {}^{h}\mathbf{a}(t) \tag{9} $$

for all t.

For the other direction, to compute the head-coincident coordinate values of velocities and accelerations from their earth-fixed values, the inverse of the linear transformation given by $R(\xi(t))$ is used:

$$ {}^{h}\mathbf{v}(t) = R(\xi(t))^{-1}\ {}^{E}\mathbf{v}(t) \tag{10} $$

$$ {}^{h}\mathbf{a}(t) = R(\xi(t))^{-1}\ {}^{E}\mathbf{a}(t) \tag{11} $$

$$ {}^{h}\boldsymbol{\omega}(t) = R(\xi(t))^{-1}\ {}^{E}\boldsymbol{\omega}(t) \tag{12} $$

$$ {}^{h}\boldsymbol{\alpha}(t) = R(\xi(t))^{-1}\ {}^{E}\boldsymbol{\alpha}(t) \tag{13} $$

for all t.

The above equations can be combined to obtain the general relation

$$ {}^{h}\mathbf{v}(t) = R(\xi(t))^{-1} \int_0^t R(\xi(\tau))\ {}^{h}\mathbf{a}(\tau)\, d\tau + {}^{h}\mathbf{v}(0) $$

relating velocities and accelerations in the head-coincident system, and when rotation is about a fixed axis, it is not difficult to show that the corresponding equation for angular motion reduces to

$$ {}^{\mathrm{h}}\boldsymbol{\omega}(t) = \int_0^t {}^{\mathrm{h}}\boldsymbol{\alpha}(\tau)\, d\tau + {}^{\mathrm{h}}\boldsymbol{\omega}(0) $$

The final equation below is not used in this paper, but it follows immediately from the equations above, and gives the trajectory of the head in earth-fixed coordinates. Writing $\mathbf{r}(t)$ for the linear position in earth-fixed coordinates of the origin of the head-coincident system, we have

$$ \mathbf{r}(t) = \int_0^t {}^{\mathrm{E}}\mathbf{v}(\tau)\, d\tau + \mathbf{r}(0) $$

$$ = \int_0^t R(\boldsymbol{\xi}(\tau))\, {}^{\mathrm{h}}\mathbf{v}(\tau)\, d\tau + \mathbf{r}(0) $$

for all t.

3. MOTIONS AND SUSTAINED MOTIONS

In this section, we present formal definitions of *motion* and *sustained motion*, and investigate the set of sustained motions. The equations of Section 2 are used extensively in proving the results here. Section 4 serves to consolidate the results in this section by giving a categorization of the set of sustained motions.

For self-motion perception, not only are linear and angular velocities and accelerations important, but orientation relative to the earth (or other host planet or moon) is relevant because of the gravitational field. The gravitational pull induces an upward reactionary force on the soles of the feet when standing on a rigid surface, for example; this z-directed force on the body is the same as that occurring during acceleration in the z direction. In this way, gravity behaves exactly like linear acceleration (this fact being Einstein's principle of equivalence), and we indicate orientation, or "attitude," relative to the gravitational field by a vector $\mathbf{a}_g$, the *attitude vector*, pointing away from the ground. The magnitude of $\mathbf{a}_g$ is equal to that of the acceleration due to gravity of a free-falling body (which is approximately 9.8 m/sec^2 at the surface of the earth).

In order to specify a subject's perspective of self-motion at a given point in time, the following formal definition is made:

Definition 2. A *motion*, referring to head motion at a given point in time, is a five-tuple of vectors in the head-coincident coordinate system of Fig. 1, the vectors for:

1. Linear velocity, denoted by **v**
2. Linear acceleration, denoted by **a**
3. Angular velocity, denoted by **ω**
4. Angular acceleration, denoted by **α**
5. Attitude, $\mathbf{a}_g$

We denote the set of all possible motions by M. In general, vectors like these are transformable between different coordinate systems and different units of measure, but we fix units of measure in advance in order to deal formally with the set of motions in the results that follow. We fix units m/sec for **v**, m/sec^2 for **a**, rad/sec for **ω**, rad/sec^2 for **α**, and g-units for $\mathbf{a}_g$ (1 g-unit ≈ 9.8 m/sec^2), so that a motion is technically a 15-dimensional vector in $R^3 \times R^3 \times R^3 \times R^3 \times R^3$ (where R denotes the set of real numbers), and M is a 15-dimensional vector space. (The results in this paper hold equally for other units of measure; the choice is arbitrary.)

Two examples of motions are given in Fig. 2. Forward travel at 800 km/hr (≈222 m/sec) while seated in a passenger jet can be specified by the appropriate vectors **v**, **a**, **ω**, **α**, and $\mathbf{a}_g$, as shown in Fig. 2a; a π/3 rad/sec counterclockwise rotation while seated on a rotating chair can be specified as shown in Fig. 2b. Each of these motions is a member of the large set M.

Both forward linear velocity and constant rotation are motions that can be sustained over time. In contrast, there are certain motions whose specified vectors cannot be sustained over time; for example, an upright position ($\mathbf{a}_g$ in the z direction) paired with forward pitch (**ω** in the y direction) causes the subject to face downward at some point, hence the resulting attitude is not in agreement with the original vector $\mathbf{a}_g$. In formally defining "sustained motion" we want as many of the vectors describing such a motion to be sustainable over time. However, because nonzero accelerations can cause velocities to change over time, it is unreasonable to expect all velocities and accelerations to remain constant; instead, we require accelerations to remain constant and allow velocities to change only in simple ways as follows:

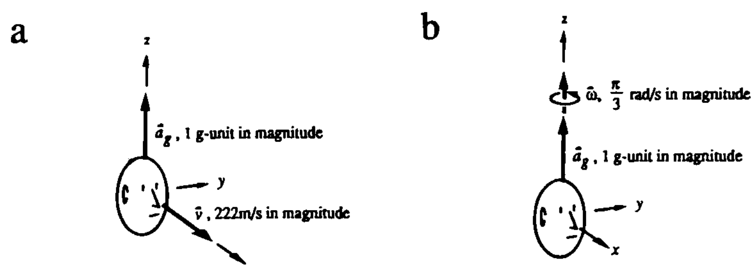

Fig. 2. Two motions belonging to the set M. (a) Forward travel at 800 km/hr (≈222 m/sec). (b) Counterclockwise rotation of π/3 rad/sec. Note that because **v**, **ω**, and $\mathbf{a}_g$ use different units of measure, no direct comparison of their lengths is intended.

Definition 3. A *sustained motion* is a motion (given by $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$) with $|\mathbf{a}_g| = 1$ g-unit, and for which the linear acceleration, the angular acceleration, and the attitude would remain equal to $\mathbf{a}$, $\boldsymbol{\alpha}$, and $\mathbf{a}_g$, respectively, throughout performance of the prescribed movement, while the linear velocity and the angular velocity would remain parallel to $\mathbf{v}$ and $\boldsymbol{\omega}$, respectively; in addition, if $\mathbf{a}$ is nonzero, then the specified $\mathbf{v}$ is required to be nonzero, and if $\boldsymbol{\alpha}$ is nonzero, then the specified $\boldsymbol{\omega}$ is required to be nonzero.

To phrase the requirements of a sustained motion in technical terms, let θ be the magnitude of the angle between $\mathbf{a}_g$ and the head's positive z axis, and let

$$\boldsymbol{\xi} = \theta \cdot \frac{\mathbf{a}_g \times \mathbf{k}}{|\mathbf{a}_g \times \mathbf{k}|}$$

($\mathbf{k}$ being the unit vector in the z direction). Notice that the $x'y'z'$ coordinate system that is in angular position $-\boldsymbol{\xi}$ with respect to the head-coincident system has an earth-horizontal $x'y'$ plane; let this coordinate system be the earth-fixed reference system for the present discussion. Now, $\mathbf{a}_g$ must have magnitude 1 g-unit, and to help fulfill "attitude remains equal to $\mathbf{a}_g$," $\boldsymbol{\omega}$ must equal $c\mathbf{a}_g$ for some real number c (possibly zero). If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{v}$ must be $\neq \mathbf{0}$, and if $\boldsymbol{\alpha} \neq \mathbf{0}$, then $\boldsymbol{\omega}$ must be $\neq \mathbf{0}$. Noting that the vectors $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{a}_g$ are not functions of time, but are single vectors, define acceleration functions of time by ${}^h\mathbf{a}(t) = \mathbf{a}$ for all t, and ${}^h\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}$ for all t. By setting ${}^h\mathbf{v}(0) = \mathbf{v}$, ${}^h\boldsymbol{\omega}(0) = \boldsymbol{\omega}$, and $\boldsymbol{\xi}(0) = \boldsymbol{\xi}$, we find that equations (1)–(13) determine the values of ${}^h\mathbf{v}(t)$, ${}^h\boldsymbol{\omega}(t)$, and $\boldsymbol{\xi}(t)$ for all t. For the given motion to be a sustained motion, there must be functions p, q: $\mathrm{R} \to \mathrm{R}$ such that, for all t,

$${}^h\mathbf{v}(t) = p(t)\mathbf{v} \quad \text{and} \quad {}^h\boldsymbol{\omega}(t) = q(t)\boldsymbol{\omega}$$

In analogy with the notation M for the set of motions, we denote the set of sustained motions by M_S.

Physiologically speaking, sustained motions are those whose components, as detected by a subject, are relatively constant. Detected linear acceleration is constant, detected angular acceleration is constant, detected linear velocity is constant in angle (relative to the subject), and detected angular velocity is constant in angle. The only reason that we do not require velocities to have constant magnitude is that the resulting set of motions would then be trivial from the viewpoint of vestibular research, and would lead to few interesting experiments. As defined presently, sustained motions form a large collection of motions, many of which are experimentally feasible; because of the constant nature of sustained motions, psychophysical measurements of self-motion perception are tractable.

We already know that certain sustained motions cause misperception or disorientation due to the leveling off of peripheral receptor response, either at resting level or another level. It is therefore reasonable to predict that calculation of the full set of sustained motions will lead to the discovery of additional motions causing misperception or disorientation. The present paper performs the required analysis, and gives a complete categorization of the set of sustained motions. Further work in Holly and McCollum (1995) builds upon the present work, leading to the prediction of entire classes of perceptually indistinguishable motions.

With the formal definitions above in mind, the investigation of the set M_S is carried out by characterizing the different types of sustained motions. Lemmas 1, 2, and 4 concentrate on those sustained motions with zero angular velocity, nonzero angular velocity with zero angular acceleration, and nonzero angular acceleration, respectively. Although the head-coincident system has certain nontraditional properties, the equations of Section 2 provide a firm foundation from which the proofs of Lemmas 1, 2, and 4 can proceed in a straightforward manner. Lemma 3 demonstrates the linear independence of a certain set of functions that appears in the proof of Lemma 4 and the mathematics is of a different nature than most in this paper. Results on linear independence of functions turn out to be indispensable in the proofs of Lemmas 2 and 4, and also underlie some reasoning in the proof of Lemma 1.

The remainder of this section consists of formal statements and proofs of the crucial results on sustained motion; these results are consolidated and restated in a less technical language in Section 4.

Lemma 1. If $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$, then the motion $(\mathbf{v}, \mathbf{a}, \boldsymbol{\omega}, \boldsymbol{\alpha}, \mathbf{a}_g)$ is a sustained motion if and only if the vectors satisfy

$$\mathbf{v} \in \mathbf{R}^3, \qquad \mathbf{a} \in \mathbf{R}^3, \qquad \boldsymbol{\omega} = \mathbf{0}, \qquad \boldsymbol{\alpha} = \mathbf{0}, \qquad \mathbf{a}_g = \mathbf{k}$$

$$\mathbf{v} \parallel \mathbf{a}, \qquad \mathbf{a} \neq \mathbf{0} \Rightarrow \mathbf{v} \neq \mathbf{0}$$

(where "$\Rightarrow$" stands for "implies").

Proof. Assume that the values $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$ are given. To determine the values that $\mathbf{v}$ and $\mathbf{a}$ must take for $(\mathbf{v}, \mathbf{a}, \boldsymbol{\omega}, \boldsymbol{\alpha}, \mathbf{a}_g)$ to be a sustained motion, we refer to the technical requirements of sustained motion. The value $\mathbf{a}_g = \mathbf{k}$ gives $\boldsymbol{\xi}(0) = \boldsymbol{\xi} = \mathbf{0}$. The values $\boldsymbol{\omega} = \mathbf{0}$ and $\boldsymbol{\alpha} = \mathbf{0}$ prescribe zero angular movement as time varies [since $\boldsymbol{\alpha}(t)$ is set to $\boldsymbol{\alpha}$ for all t], so equation (3) immediately gives $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(0) = \mathbf{0}$.

The matrix $R(\boldsymbol{\xi}(t))$ is thus the identity matrix (for all t), and we obtain ${}^{\mathrm{E}}\mathbf{v}(t) = {}^{\mathrm{h}}\mathbf{v}(t)$ and ${}^{\mathrm{E}}\mathbf{a}(t) = {}^{\mathrm{h}}\mathbf{a}(t) = \mathbf{a}$ for all t, as well as ${}^{\mathrm{E}}\mathbf{v}(0) = {}^{\mathrm{h}}\mathbf{v}(0) = \mathbf{v}$. Next, we use equation (8) to see that for all t,

$$
\begin{aligned}
{}^{\mathrm{h}}\mathbf{v}(t) = {}^{\mathrm{E}}\mathbf{v}(t) &= \int_0^t {}^{\mathrm{E}}\mathbf{a}(\tau)\, d\tau + {}^{\mathrm{E}}\mathbf{v}(0) \\
&= \int_0^t \mathbf{a}\, d\tau + \mathbf{v} \\
&= \mathbf{a}t + \mathbf{v}
\end{aligned}
$$

Given this formula, in order for ${}^{\mathrm{h}}\mathbf{v}(t)$ to be parallel to $\mathbf{v}$ for all t, $\mathbf{a}$ and $\mathbf{v}$ must be parallel. (In the case $\mathbf{a} = \mathbf{0}$, every vector $\mathbf{v}$ is parallel to $\mathbf{a}$.)

Therefore, one requirement is that $\mathbf{v} \parallel \mathbf{a}$. Under the present conditions, the definition of sustained motions gives no additional restrictions other than $\mathbf{a} \neq \mathbf{0} \Rightarrow \mathbf{v} \neq \mathbf{0}$, concluding the proof. QED

For Lemmas 2 and 4 that follow, the convention is made that

$$
\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}
$$

Lemma 2. If $\boldsymbol{\omega} \neq \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$, then the motion $(\mathbf{v}, \mathbf{a}, \boldsymbol{\omega}, \boldsymbol{\alpha}, \mathbf{a}_g)$ is a sustained motion if and only if the vectors satisfy

$$
\boldsymbol{\omega} = \omega\mathbf{k} \quad \text{for some nonzero } \omega \in \mathrm{R}, \qquad \mathbf{a} = \mathbf{0}, \qquad \boldsymbol{\alpha}_g = \mathbf{k}
$$

and either (i)

$$
a_x = a_y = 0, \quad 0 \neq a_z \in \mathrm{R}, \qquad v_x = v_y = 0, \quad 0 \neq v_z \in \mathrm{R}
$$

or (ii)

$$
a_x, a_y \in \mathrm{R}, \quad a_z = 0, \qquad v_x = \frac{a_y}{\omega}, \quad v_y = -\frac{a_x}{\omega}, \quad v_z \in \mathrm{R}
$$

Proof. Assuming that the values $\boldsymbol{\alpha} = \mathbf{0}$ and $\mathbf{a}_g = \mathbf{k}$ are given, the definition of sustained motion requires that $\boldsymbol{\omega} = c\mathbf{a}_g$ for some real number c. Assuming for this lemma that $\boldsymbol{\omega} \neq \mathbf{0}$, we write $\boldsymbol{\omega} = \omega\mathbf{k} = \omega\mathbf{a}_g$, where $\omega \neq 0$.

Given $\boldsymbol{\alpha}$, $\mathbf{a}_g$, and $\boldsymbol{\omega}$ as described, the values that $\mathbf{v}$ and $\mathbf{a}$ must take for $(\mathbf{v}, \mathbf{a}, \boldsymbol{\omega}, \boldsymbol{\alpha}, \mathbf{a}_g)$ to be a sustained motion are determined by using the technical requirements of sustained motion. The value $\mathbf{a}_g = \mathbf{k}$ gives $\boldsymbol{\xi}(0) = \boldsymbol{\xi} = \mathbf{0}$, making $R(\boldsymbol{\xi}(0))$ the identity matrix. Setting ${}^{\mathrm{h}}\boldsymbol{\omega}(0) = \boldsymbol{\omega}$ and ${}^{\mathrm{h}}\mathbf{v}(0) = \mathbf{v}$, we obtain from equations (5) and (7) with $t = 0$

$$^{E}\boldsymbol{\omega}(0) = {}^{h}\boldsymbol{\omega}(0) = \boldsymbol{\omega} = \omega\mathbf{k}$$
$$^{E}\mathbf{v}(0) = {}^{h}\mathbf{v}(0) = \mathbf{v}$$

In order to analyze the situation as time varies, we must know the form of the linear transformation $R(\boldsymbol{\xi}(t))$, which depends on $\boldsymbol{\xi}(t)$. To obtain $\boldsymbol{\xi}(t)$ in general, we want to use equation (3). But first: Note that since all angular movement is about a vertical axis, the head-coincident coordinate system and the earth-fixed coordinate system always have their z axes aligned. This means that for all values of t, the transformations from head-coincident to earth-fixed coordinates and vice versa leave z-directed vectors unchanged. With this in mind, setting $^{h}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha} = \mathbf{0}$ for all t, we see that

$$^{E}\boldsymbol{\alpha}(t) = {}^{h}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha} = \mathbf{0}$$

for all t, by equation (6) in combination with the fact that z-directed vectors (including the trivial zero vector) remain unchanged by the linear transformation.

Proceeding toward the calculation of $\boldsymbol{\xi}(t)$ and $R(\boldsymbol{\xi}(t))$, equation (4) gives $^{E}\boldsymbol{\omega}(t) = {}^{E}\boldsymbol{\omega}(0) = \omega\mathbf{k}$ for all t since $^{E}\boldsymbol{\alpha}(\tau) = \mathbf{0}$ for all τ. Then equation (3) gives

$$\begin{aligned}\boldsymbol{\xi}(t) &= \int_0^t {}^{E}\boldsymbol{\omega}(\tau)\, d\tau + \boldsymbol{\xi}(0)\\ &= \int_0^t \omega\mathbf{k}\, d\tau + \mathbf{0}\\ &= \omega t\mathbf{k}\end{aligned}$$

for all t since rotation is about a fixed axis. Substituting $\omega t\mathbf{k}$ for $\boldsymbol{\xi}$ in equations (2) and (1) (with $\xi_x = 0$, $\xi_y = 0$, $\xi_z = \omega t$, and $|\boldsymbol{\xi}| = \omega$), we obtain for the matrix $R(\boldsymbol{\xi}(t))$

$$R(\boldsymbol{\xi}(t)) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0\\ \sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1\end{pmatrix} \tag{14}$$

for all t.

The trick now is to find which values of $\mathbf{v}$ and $\mathbf{a}$ cause $^{h}\mathbf{v}(t)$ to remain parallel to $\mathbf{v}$ as time t varies. Setting $^{h}\mathbf{a}(t) = \mathbf{a}$ for all t, as suggested in the definition of sustained motion, the form of $^{E}\mathbf{a}(t)$ can be obtained from equations (9) and (14):

$$^{E}\mathbf{a}(t) = R(\boldsymbol{\xi}(t))\, {}^{h}\mathbf{a}(t) = \begin{pmatrix} a_x\cos\omega t - a_y\sin\omega t\\ a_x\sin\omega t + a_y\cos\omega t\\ a_z\end{pmatrix}$$

for all t.

The use of equations (8) and (10) in succession results in an expression for ${}^{h}\mathbf{v}(t)$ which can then be analyzed for "parallelism with **v**." First, equation (8) gives

$$
{}^{E}\mathbf{v}(t) = \int_0^t {}^{E}\mathbf{a}(\tau)\, d\tau + {}^{E}\mathbf{v}(0) = \begin{pmatrix} v_x + \dfrac{a_x}{\omega} \sin \omega t + \dfrac{a_y}{\omega} \cos \omega t - \dfrac{a_y}{\omega} \\ v_y - \dfrac{a_x}{\omega} \cos \omega t + \dfrac{a_y}{\omega} \sin \omega t + \dfrac{a_x}{\omega} \\ v_z + a_z t \end{pmatrix}
$$

for all t. To use equation (10) next, note that the inverse of $R(\xi(t))$ is just the transpose of $R(\xi(t))$, and the result of the calculation is

$$
{}^{h}\mathbf{v}(t) = R(\xi(t))^{-1}\, {}^{E}\mathbf{v}(t) = \begin{pmatrix} \dfrac{a_y}{\omega} + \left(v_x - \dfrac{a_y}{\omega}\right) \cos \omega t + \left(v_y + \dfrac{a_x}{\omega}\right) \sin \omega t \\ -\dfrac{a_x}{\omega} + \left(v_y + \dfrac{a_x}{\omega}\right) \cos \omega t + \left(-v_x + \dfrac{a_y}{\omega}\right) \sin \omega t \\ v_z + a_z t \end{pmatrix} \tag{15}
$$

for all t. What remains is to determine the values of v_x, v_y, v_z, a_x, a_y, and a_z for which the three components of the vector ${}^{h}\mathbf{v}(t)$ in (15) are in direct proportion to one another as functions of t on R (the function identically equal to zero being in direct proportion to every real-valued function on R). There are two cases to analyze.

Case $a_z \neq 0$. For the first component of the vector ${}^{h}\mathbf{v}(t)$ in equation (15) to be directly proportional to the third component $v_z + a_z t$, where $a_z \neq 0$, says that there exists a nonzero constant $c \in$ R such that

$$
v_z + a_z t = c\left(\frac{a_y}{\omega} + \left(v_x - \frac{a_y}{\omega}\right) \cos \omega t + \left(v_y + \frac{a_x}{\omega}\right) \sin \omega t\right)
$$

for all t. Therefore, the terms a_y/ω, $(v_x - a_y/\omega)$, and $(v_y + a_x/\omega)$ must all equal zero; this follows from the fact that for every nonzero ω, the set $\{1, t, \cos \omega t, \sin \omega t\}$ of functions of $t \in$ R is a linearly independent set of functions on R. By a similar analysis requiring the second and third components of ${}^{h}\mathbf{v}(t)$ to be directly proportional, we also see that $-a_x/\omega$ must equal zero. In summary, the only permitted values under the given assumptions when $a_z \neq 0$ are

$$
a_x = a_y = 0, \quad 0 \neq a_z \in \mathrm{R}, \qquad v_x = v_y = 0, \quad 0 \neq v_z \in \mathrm{R}
$$

where $v_z \neq 0$ is required by the definition of sustained motion since $\mathbf{a} \neq \mathbf{0}$

here. These permitted values are exactly those of (i) in the statement of the lemma.

Case $a_z = 0$. Checking the direct proportionality of the first two components of ${}^{h}\mathbf{v}(t)$ in equation (15), suppose that

$$\frac{a_y}{\omega} + \left(v_x - \frac{a_y}{\omega}\right)\cos\omega t + \left(v_y + \frac{a_x}{\omega}\right)\sin\omega t$$
$$= c\left(-\frac{a_x}{\omega} + \left(v_y + \frac{a_x}{\omega}\right)\cos\omega t + \left(-v_x + \frac{a_y}{\omega}\right)\sin\omega t\right) \tag{16}$$

for some $c \in \mathbf{R}$. Using the fact that the set $\{1, \cos\omega t, \sin\omega t\}$ of functions of $t \in \mathbf{R}$ is a linearly independent set of functions on R, and setting the corresponding coefficients in (16) equal to one another, we obtain that the resulting system of linear equations reduces to

$$a_y = c(-a_x), \qquad v_x = \frac{a_y}{\omega}, \qquad v_y = -\frac{a_x}{\omega} \tag{17}$$

Now, just in case the first component of ${}^{h}\mathbf{v}(t)$ can be identically zero while the second is nonzero, an analogous calculation must be done, starting with the supposition (second) $= d$(first) for some $d \in \mathbf{R}$. The resulting relationships between a_x, a_y, v_x, and v_y happen to be the same as in (17), except that $-a_x = d(a_y)$ instead of $a_y = c(-a_x)$. Since the constants c and d in the equations can be any real numbers, we conclude that

$$a_x, a_y \in \mathbf{R}, \quad a_z = 0, \qquad v_x = \frac{a_y}{\omega}, \quad v_y = -\frac{a_x}{\omega}, \quad v_z \in \mathbf{R}$$

are the permitted values for a sustained motion when $a_z = 0$, under the given assumptions. From the definition of sustained motion, we see that there are no other restrictions. These permitted values are exactly those of (ii) in the statement of the lemma.

In summary, a sustained motion with $\boldsymbol{\omega} \neq \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$ must have values $\mathbf{v}$ and $\mathbf{a}$ given by (i) if $a_z \neq 0$ and by (ii) if $a_z = 0$. QED

Before proceeding with the investigation of M_S, Lemma 3 is necessary (to be used in the proof of Lemma 4).

Lemma 3. For every $\omega \in \mathbf{R}$ and every nonzero $\alpha \in \mathbf{R}$, the set

$$\left\{\cos\left(\frac{1}{2}\alpha t^2 + \omega t\right),\right.$$
$$\sin\left(\frac{1}{2}\alpha t^2 + \omega t\right),$$

$$\cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau + \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right)$$

$$\int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau,$$

$$\cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau - \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right)$$

$$\left.\int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau\right\}$$

of functions of $t \in \mathrm{R}$ is a linearly independent set of functions on R.

Proof. We give the proof for the case α and ω nonnegative; the proofs for the other cases are similar. Fix $\alpha > 0$ and $\omega \geq 0$. Suppose that

$$\begin{aligned} & b_1 \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) + b_2 \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) \\ & + b_3\left(\cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau \right. \\ & \left. + \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau\right) \\ & + b_4\left(\cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau \right. \\ & \left. - \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) \int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau\right) \equiv 0 \end{aligned} \tag{18}$$

for some $b_1, b_2, b_3, b_4 \in \mathrm{R}$ (where "$\equiv 0$" means "$= 0$ for all $t \in \mathrm{R}$"). The goal is to show that b_1, b_2, b_3, and b_4 must all be zero.

By substituting $t = 0$ into equation (18), all terms become zero except the first, so b_1 must be zero. Now let

$$t_n = \frac{\sqrt{\omega^2 + \alpha n\pi} - \omega}{\alpha}, \qquad n = 1, 2, 3$$

so that $(\frac{1}{2}\alpha t_n^2 + \omega t_n) = n\pi/2$ for $n = 1, 2, 3$. Knowing that $b_1 = 0$, we substitute successively $t = t_1$, $t = t_2$, and $t = t_3$ into equation (18) and make the change of variable $\sigma = \frac{1}{2}\alpha\tau^2 + \omega\tau$ in the integrals to get the three equations

$$b_2 + b_3\left(\int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \sin \sigma \, d\sigma\right) - b_4\left(\int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma\right) = 0$$
$$-b_3\left(\int_0^{\pi} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma\right) - b_4\left(\int_0^{\pi} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \sin \sigma \, d\sigma\right) = 0$$
$$-b_2 - b_3\left(\int_0^{3\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \sin \sigma \, d\sigma\right) + b_4\left(\int_0^{3\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma\right) = 0 \tag{19}$$

Let

$$I_n = \int_0^{n\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma,$$
$$J_n = \int_0^{n\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \sin \sigma \, d\sigma,$$
$$n = 1, 2, 3$$

The system (19) is equivalent to the system

$$b_2 + b_3 J_1 - b_4 I_1 = 0$$
$$-b_3 I_2 - b_4 J_2 = 0$$
$$b_4(I_2(I_1 - I_3) + J_2(J_1 - J_3)) = 0$$

so to show that $b_2 = b_3 = b_4 = 0$, it suffices to show that $(I_2(I_1 - I_3) + J_2(J_1 - J_3)) \neq 0$ and that $I_2 \neq 0$.

We know that $J_2 > 0$ since $\sin \sigma > 0$ for $\sigma \in (0, \pi)$, and that $(I_1 - I_3) > 0$ since $\cos \sigma < 0$ for $\sigma \in (\pi/2, 3\pi/2)$. We claim that $I_2 > 0$ and $(J_1 - J_3) > 0$ also. Note that $1/(\omega^2 + 2\alpha\sigma)^{1/2}$ is a strictly positive, strictly increasing function of σ for $\sigma > 0$ (since $\alpha > 0$). Using this and the fact that $\cos \sigma > 0$ for $\sigma \in (0, \pi/2)$ and $\cos \sigma < 0$ for $\sigma \in (\pi/2, \pi)$, we obtain

$$I_2 = \int_0^{\pi} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma$$
$$= \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma + \int_{\pi/2}^{\pi} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma$$
$$> \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma + \int_{\pi/2}^{\pi} \frac{1}{\sqrt{\omega^2 + 2\alpha(\pi/2)}} \cos \sigma \, d\sigma$$
$$= \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma - \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha(\pi/2)}} \cos \sigma \, d\sigma$$

$$> \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma - \int_0^{\pi/2} \frac{1}{\sqrt{\omega^2 + 2\alpha\sigma}} \cos \sigma \, d\sigma$$

$$= 0$$

A similar trick can be applied to $(J_1 - J_3)$ to show that $(J_1 - J_3) > 0$.

Therefore, $(I_2(I_1 - I_3) + J_2(J_1 - J_3)) > 0$ and $I_2 > 0$, implying that $b_2 = b_3 = b_4 = 0$. QED

The final lemma deals with sustained motions that have nonzero angular acceleration.

Lemma 4. If $\boldsymbol{\alpha} \neq \mathbf{0}$ and $\mathbf{a}_g = \mathbf{k}$, then the motion $(\mathbf{v}, \mathbf{a}, \boldsymbol{\omega}, \boldsymbol{\alpha}, \mathbf{a}_g)$ is a sustained motion if and only if the vectors satisfy

$$\boldsymbol{\omega} = \omega\mathbf{k} \qquad \text{for some nonzero } \omega \in \mathrm{R}$$

$$\boldsymbol{\alpha} = \alpha\mathbf{k} \qquad \text{for some nonzero } \alpha \in \mathrm{R}$$

$$\mathbf{a}_g = \mathbf{k}, \quad a_x = a_y = 0, \quad a_z \in \mathrm{R}, \qquad v_x = v_y = 0, \quad v_z \in \mathrm{R}$$

$$a_z \neq 0 \Rightarrow v_z \neq 0$$

Proof. Assuming that $\mathbf{a}_g = \mathbf{k}$, and that a nonzero $\boldsymbol{\alpha}$ is given, the definition of sustained motion requires $\boldsymbol{\omega} \neq \mathbf{0}$, and in particular, $\boldsymbol{\omega} = \omega\mathbf{k}$ for some real number $\omega \neq 0$.

Continuing with the definition of sustained motion, we have that the value $\mathbf{a}_g = \mathbf{k}$ gives $\boldsymbol{\xi}(0) = \boldsymbol{\xi} = \mathbf{0}$, making $R(\boldsymbol{\xi}(0))$ the identity matrix. Set ${}^{\mathrm{h}}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}$ for all t, set ${}^{\mathrm{h}}\boldsymbol{\omega}(0) = \boldsymbol{\omega}$, and set ${}^{\mathrm{h}}\mathbf{v}(0) = \mathbf{v}$. Then equations (5)–(7) with $t = 0$ give

$${}^{\mathrm{E}}\boldsymbol{\alpha}(0) = {}^{\mathrm{h}}\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}$$

$${}^{\mathrm{E}}\boldsymbol{\omega}(0) = {}^{\mathrm{h}}\boldsymbol{\omega}(0) = \boldsymbol{\omega} = \omega\mathbf{k}$$

$${}^{\mathrm{E}}\mathbf{v}(0) = {}^{\mathrm{h}}\mathbf{v}(0) = \mathbf{v}$$

respectively. Equation (4) shows ${}^{\mathrm{E}}\boldsymbol{\omega}(t)$ to be

$$\begin{aligned} {}^{\mathrm{E}}\boldsymbol{\omega}(t) &= \int_0^t {}^{\mathrm{E}}\boldsymbol{\alpha}(\tau)\, d\tau + {}^{\mathrm{E}}\boldsymbol{\omega}(0) \\ &= \int_0^t {}^{\mathrm{E}}\boldsymbol{\alpha}(\tau)\, d\tau + \omega\mathbf{k} \end{aligned}$$

for all t since rotation is about a fixed axis. Because ${}^{\mathrm{E}}\boldsymbol{\omega}(t)$ must be parallel to $\omega\mathbf{k}$ for all t, and ${}^{\mathrm{E}}\boldsymbol{\alpha}(\tau)$ is a continuous function of τ, we conclude that ${}^{\mathrm{E}}\boldsymbol{\alpha}(0)$ must have no nonzero off-vertical (away from $\mathbf{k}$) component. In other words, ${}^{E}\boldsymbol{\alpha}(0) = \alpha\mathbf{k}$ for some $\alpha \in \mathbf{R}$, this lemma requiring $\alpha \neq 0$. Since ${}^{h}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}$ for all t, we have

$$ {}^{h}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha} = {}^{h}\boldsymbol{\alpha}(0) = \alpha\mathbf{k} $$

for all t.

Using this expression for ${}^{h}\boldsymbol{\alpha}(t)$, we can obtain expressions for ${}^{E}\boldsymbol{\alpha}(t)$, ${}^{E}\boldsymbol{\omega}(t)$, $\boldsymbol{\xi}(t)$, and $R(\boldsymbol{\xi}(t))$ in succession. First, note that all angular movement is about a vertical axis, so the head-coincident coordinate system and the earth-fixed coordinate system always have their z axes aligned. This means that for all values of t, the transformations from head-coincident to earth-fixed coordinates and vice versa leave z-directed vectors unchanged. With this in mind, we see that

$$ {}^{E}\boldsymbol{\alpha}(t) = {}^{h}\boldsymbol{\alpha}(t) = \alpha\mathbf{k} $$

for all t, since the z-directed vector $\alpha\mathbf{k}$ remains unchanged by the linear transformation in equation (6).

Equations (4) and (3) result in

$$ {}^{E}\boldsymbol{\omega}(t) = (\alpha t + \omega)\mathbf{k} $$
$$ \boldsymbol{\xi}(t) = \left(\frac{1}{2}\alpha t^2 + \omega t\right)\mathbf{k} $$

for all t. Substituting $(\frac{1}{2}\alpha t^2 + \omega t)\mathbf{k}$ for $\boldsymbol{\xi}$ in (2) and (1), we find for the matrix $R(\boldsymbol{\xi}(t))$

$$ R(\boldsymbol{\xi}(t)) = \begin{pmatrix} \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) & -\sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) & 0 \\ \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) & \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (20) $$

for all t.

With an expression for $R(\boldsymbol{\xi}(t))$, we can concentrate on finding which values of $\mathbf{v}$ and $\mathbf{a}$ make the present motion a sustained motion, by finding which values of $\mathbf{v}$ and $\mathbf{a}$ cause ${}^{h}\mathbf{v}(t)$ to remain parallel to $\mathbf{v}$ as time t varies. Setting ${}^{h}\mathbf{a}(t) = \mathbf{a}$ for all t, we have from equations (9) and (20)

$$ {}^{E}\mathbf{a}(t) = R(\boldsymbol{\xi}(t))\,{}^{h}\mathbf{a}(t) = \begin{pmatrix} a_x \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) - a_y \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) \\ a_x \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right) + a_y \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right) \\ a_z \end{pmatrix} $$

for all t. Equation (8) gives

$$
{}^{\mathrm{E}}\mathbf{v}(t) = \int_0^t {}^{\mathrm{E}}\mathbf{a}(\tau)\, d\tau + {}^{\mathrm{E}}\mathbf{v}(0)
$$

$$
= \begin{pmatrix} v_x + a_x \int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau - a_y \int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau \\ v_y + a_x \int_0^t \sin\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau + a_y \int_0^t \cos\left(\frac{1}{2}\alpha\tau^2 + \omega\tau\right) d\tau \\ v_z + a_z t \end{pmatrix}
$$

for all t. To use equation (10) next, note that the inverse of $R(\boldsymbol{\xi}(t))$ is just the transpose of $R(\boldsymbol{\xi}(t))$. Letting

$$
f(t) = \cos\left(\frac{1}{2}\alpha t^2 + \omega t\right), \qquad g(t) = \sin\left(\frac{1}{2}\alpha t^2 + \omega t\right)
$$

for all t, we obtain from equation (10)

$$
{}^{\mathrm{h}}\mathbf{v}(t) = \begin{pmatrix} v_x f(t) + v_y g(t) + a_x\left(f(t)\int_0^t f(\tau)\, d\tau + g(t)\int_0^t g(\tau)\, d\tau\right) \\ \quad - a_y\left(f(t)\int_0^t g(\tau)\, d\tau - g(t)\int_0^t f(\tau)\, d\tau\right) \\ v_y f(t) - v_x g(t) + a_y\left(f(t)\int_0^t f(\tau)\, d\tau + g(t)\int_0^t g(\tau)\, d\tau\right) \\ \quad + a_x\left(f(t)\int_0^t g(\tau)\, d\tau - g(t)\int_0^t f(\tau)\, d\tau\right) \\ v_z + a_z t \end{pmatrix} \tag{21}
$$

for all t.

What remains is to determine the values of v_x, v_y, v_z, a_x, a_y, and a_z for which the three components of the vector in (21) are in direct proportion to one another as functions of t on R. By Lemma 3, the set

$$
\left\{f(t), g(t), f(t)\int_0^t f(\tau)\, d\tau + g(t)\int_0^t g(\tau)\, d\tau, \right.
$$

$$
\left. f(t)\int_0^t g(\tau)\, d\tau - g(t)\int_0^t f(\tau)\, d\tau\right\}
$$

of functions of $t \in \mathrm{R}$ is a linearly independent set of functions on R. Therefore, the first two components of ${}^{\mathrm{h}}\mathrm{v}(t)$ in equation (21) are in direct proportion as functions on R only if $v_z = v_y = a_x = a_y = 0$ or if there is a constant $c \in$ R such that $v_y = cv_x$, $-v_x = cv_y$, $a_y = ca_x$, and $a_x = c(-a_y)$. In the latter case, $v_x = v_y = a_x = a_y = 0$ is still the only solution.

Having determined that $v_x = v_y = a_x = a_y = 0$ is required, the values of v_z and a_z in the third component of ${}^h\mathbf{v}(t)$ in equation (21) are fairly unrestricted by the definition of sustained motion. The only remaining requirement is that v_z be nonzero whenever a_z is nonzero. These requirements, along with the expressions obtained for $\boldsymbol{\omega}$ [$= {}^h\boldsymbol{\omega}(0)$] and $\boldsymbol{\alpha}$, are exactly those in the statement of the lemma. QED

4. CATEGORIZATION OF SUSTAINED MOTIONS

The results of Section 3 can now be consolidated to give a complete description of M_S, the set of sustained motions. The sustained motions fall into natural categories, and the description of M_S is divided into those categories, while the proof that this fully describes M_S is presented after the category listing and examples below. In each of the following descriptions, any velocity or acceleration not mentioned and not arising from those mentioned is assumed to be zero.

Categories of Sustained Motion

1. *Fixed position.*
2. *Linear velocity.* This refers to linear velocity that is nonzero and constant.
3. *Linear acceleration.* Nonzero acceleration is constant and parallel to the nonzero linear velocity.
4. *Angular velocity.* Nonzero constant angular velocity occurs about an earth-vertical axis. The vertical axis is required in order for the attitude vector to remain fixed relative to the head. (The head itself need not be vertical.) Linear velocity may arise, but only in the form of tangential velocity, as may linear acceleration only in the form of centripetal acceleration.
5. *Angular and linear velocity.* Nonzero constant angular velocity occurs about an earth-vertical axis, and the linear velocity has a nonzero earth-vertical component in addition to a possible tangential component due to angular velocity at a nonzero radius. (Linear acceleration may arise in the form of centripetal acceleration.)
6. *Angular velocity and linear acceleration.* Nonzero constant angular velocity occurs about an earth-vertical axis through the head, and nonzero linear acceleration is earth-vertical and constant (causing nonzero linear velocity).

7. *Angular acceleration.* Nonzero constant angular acceleration occurs about an earth-vertical axis through the head (causing nonzero angular velocity).
8. *Angular acceleration and linear velocity.* Nonzero constant angular acceleration occurs about an earth-vertical axis through the head (causing nonzero angular velocity), and linear velocity is nonzero, constant, and earth-vertical.
9. *Angular and linear acceleration.* Nonzero constant angular acceleration occurs about an earth-vertical axis through the head (causing nonzero angular velocity), and linear acceleration is nonzero, constant, and earth-vertical (causing nonzero linear velocity).

Table I gives a summary of the permitted values and the interdependencies of $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{a}_g$ for each of the nine categories of sustained motion.

Examples.

A. Standing stationary on the ground is a member of category 1. Lying on one's back is also a member of category 1, as is hanging upside down.
B. Traveling straight forward at 800 km/hr in a jet is a member of category 2.
C. A constant-velocity version of the graveyard spiral—sometimes called the spiral dive—is a member of category 5. (Actually, since there are many possible velocities, turning radii, and speeds of descent, there are technically many different spiral dives that are members of category 5.)
D. Sitting in a chair rotating at $\pi/3$ rad/sec is a member of category 4.
E. Sitting in a chair rotating at $\pi/3$ rad/sec and increasing speed at $\pi/12$ rad/sec^2 is a member of category 7.

Theorem. The set M_S consists exactly of the motions in categories 1–9.

Proof. Using $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$, as usual, to specify a motion, first notice that Lemmas 1, 2, and 4 deal only with upright attitude, that is, $\mathbf{a}_g = \mathbf{k}$. To deal with attitude other than upright, i.e., with $|\mathbf{a}_g| = 1$ and $\mathbf{a}_g \neq \mathbf{k}$, a simple preliminary transformation is necessary. To put the explanation in nontechnical terms, imagine the subject's head oriented according to $\mathbf{a}_g$ (not necessarily upright). Now, picture an imaginary *upright* head occupying the same physical space as the subject's head. Attach the two heads (subject's and imaginary upright) firmly together. At this point, use the appropriate lemma(s) on the imaginary head to see what kinds of motions are sustained motions of the imaginary head. Finally, assuming that the subject's head is fixed as described to the imaginary head, the subject's head will "perform"

Table I. Permitted Values of Linear and Angular Velocities and Accelerations in the Nine Categories of Sustained Motion

	Attitude $\mathbf{a}_g$	Linear velocity $\mathbf{v}$	Linear acceleration $\mathbf{a}$	Angular velocity $\boldsymbol{\omega}$	Angular acceleration $\boldsymbol{\alpha}$
1. Fixed position	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	**0**	**0**	**0**	**0**
2. Linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Unrestricted, $\neq\mathbf{0}$	**0**	**0**	**0**
3. Linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Unrestricted, $\neq\mathbf{0}$	Parallel to $\mathbf{v}$	**0**	**0**
4. Angular velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Depends on $\boldsymbol{\omega}$ and radius of rotation	Depends on $\boldsymbol{\omega}$ and radius of rotation	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	**0**
5. Angular and linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Partially depends on angular motion, $\neq\mathbf{0}$	Depends on $\boldsymbol{\omega}$ and radius of rotation	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	**0**
6. Angular velocity and linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	**0**
7. Angular acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	**0**	**0**	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$
8. Angular acceleration and linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	**0**	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$
9. Angular and linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$

a sustained motion exactly when the imaginary head "performs" a sustained motion. All of this can be made formal, but the conclusion is that Lemmas 1, 2, and 4 work for any attitude by substituting the desired $\mathbf{a}_g$ for $\mathbf{k}$. (Incidentally, another way to approach the problem is to use a different head-coincident coordinate system if the subject is not upright.)

Now, Lemma 1 shows that all sustained motions with $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$ have $\mathbf{v} \parallel \mathbf{a}$ along with $\mathbf{a} \neq \mathbf{0} \Rightarrow \mathbf{v} \neq \mathbf{0}$. Categories 1–3 consist exactly of these motions and the corresponding motions with nonupright attitude.

Lemma 2 shows that some sustained motions with $\boldsymbol{\omega} \neq \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$ have nonzero constant angular velocity with a vertical axis, along with nonzero vertical linear acceleration and velocity (a_z, $v_z \neq 0$). Category 6 consists of exactly these motions and the corresponding motions with nonupright attitude. Lemma 2 shows that the remaining sustained motions with $\boldsymbol{\omega} \neq \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{a}_g = \mathbf{k}$ have nonzero constant angular velocity with a vertical axis, along with the horizontal components of $\mathbf{a}$ and $\mathbf{v}$ matching so that the movement is circular with $\mathbf{a}$ being the centripetal acceleration, while an additional vertical component of linear velocity is allowed. Categories 4 and 5 consist exactly of these motions and the corresponding motions with nonupright attitude.

Lemma 4 shows that all sustained motions with $\boldsymbol{\alpha} \neq \mathbf{0}$ and $\mathbf{a}_g = \mathbf{k}$ have nonzero constant angular acceleration with a vertical axis, nonzero angular velocity with a vertical axis, and possible vertical linear velocity or vertical linear velocity and acceleration. Categories 7–9 consist of exactly these motions and the corresponding motions with nonupright attitude.

Therefore, all sustained motions fall into categories 1–9 and we also see that categories 1–9 contain no other motions. QED

5. DISCUSSION

We have taken the subject's viewpoint in defining *subject-coincident coordinate systems* for self-motion, and in defining and classifying *sustained motions*. These subject-coincident systems, by incorporating the fact that the sensory receptors of an organism move with the organism, have unusual properties including the fact that acceleration is not the time derivative of velocity. The essential equations of motions are presented, as derived from first principles. These equations are necessary in the development of a classification of sustained motions, and after formally defining the concept of a sustained motion, we demonstrate that the set of sustained motions can be described by nine distinct categories which cover the various possible combinations of linear velocity, linear acceleration, angular velocity, and angular acceleration.

An important ingredient in the equations of motion is the linear transformation that maps between subject-coincident coordinate systems and earth-fixed coordinate systems [see equation (1)]. While linear systems theory has been used in developing models of human spatial orientation such as that of Borah *et al.* (1988), the special properties of a subject-coincident coordinate system have yet to be made explicit. The incorporation of this physiologically natural type of coordinate system into models of human spatial orientation and self-motion perception may lead to interesting results. At the same time, the neuroscientist may ask whether a linear transformation such as that described in equation (1) takes place in the brain; transformations of various types are known to take place in several parts of the brain, and a transformation between subject-coincident coordinates and earth-fixed coordinates is an obvious candidate. [For an overview of the subject and of neuroscience in general, see Kandel *et al.*, (1991).]

Using subject-coincident systems, sustained motions are those for which the linear acceleration, angular acceleration, and attitude (orientation with respect to gravity) are sustainable over time while the linear and angular velocities change only in magnitude. The spiral dive in an airplane is an example of such a motion, whereas tumbling head-over-heels is an example where attitude is not sustained. The complete set of sustained motions is determined by using the equations of Section 2, resulting in the nine categories described in Section 4.

Several observations can be made about the sustained motion categories. First, we observe that attitude is in some sense immaterial. More precisely, if a motion with a particular attitude is sustainable, then that same movement over the earth in any other attitude is also sustainable; hence, a pilot with head tilted to the side or leaning back is just as susceptible to the dangers of the graveyard spiral as a pilot sitting straight upright.

Another observation is that a number of different combinations of linear and angular movement are possible in sustained motion. In particular, linear and angular acceleration may both be zero, both be nonzero, or either may be zero while the other is nonzero. One clear restriction through all categories of sustained motion, however, is that any angular movement must have an earth-vertical axis. In addition, if angular velocity is accompanied by nonzero angular acceleration, then the radius of rotation must be zero (the rotation being on-axis). For example, turn at a constant radius in a car is sustainable, but if there is accompanying acceleration, then the motion is not a sustained motion.

In order to form a global picture of self-motion perception, many pieces are necessary. Experimental research has investigated certain classes of sustained motion [see review by Guedry (1974)], and it would be fruitful to investigate others. By definition, and by the nature of human physiology, sustained motions cause the vestibular system to saturate or adapt due to peripheral receptor response leveling off. In adaptation, the vestibular system stops signaling movement, such as during constant-velocity rotation or during constant-speed flight. On the other hand, constant angular *acceleration* may cause saturation of the vestibular system leading to dizziness and motion sickness. Both adaptation and saturation can lead to disorientation and inaccurate perception of self-motion.

As in all scientific work, an obvious inadequacy here lies in what is *not* explained. Now that sustained motions are characterized, what perceptions of self-motion are associated with them? How accurate is perception under sustained conditions? When might one motion be misinterpreted as another motion? In attacking the latter question, recent work (Holly and McCollum, 1995) using subject-coincident coordinate systems has identified and described entire classes of perceptually indistinguishable sustained motions, both when visual input is available and when it is not available. This work can help to identify possible problematic situations in air- and spaceflight, as well as in movement on the surface of the earth. (Skiers, snowboarders, and skaters have some practical experience in these matters; the author recently heard about a person snowboarding under low-visibility conditions down the side of Mt. Hood, suddenly realizing to his surprise that he was facing backward.)

With a firm foundation in subject-coincident coordinate systems, self-motion research can proceed more easily in the direction of complex and changing motion. Until recently, most motion stimuli in experiments have been simple enough to describe with a few parameters. However, an occasional experiment such as those of Cohen *et al.* (1973) and Guedry *et al.* (1992) has needed a more complicated description of motion, so "acceleration profiles" have been used. These are, in fact, descriptions of motion within a subject-coincident system. Thus, subject-coincident systems are not only consistent with previous research, but provide a way to discuss both accelerations and velocities, relate the two by means of equations of motion that are custom fit to self-motion research, and give a standard by which to discuss and compare the wide variety of motions in experimental research.

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